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If a Picture Speaks A Thousand Words, what might One Mathematical Symbol Say?

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Shaping minds, shaping the world

Symbols & the language of maths

Reading, writing, recognising and understanding symbols underpins all mathematics topics.

A myriad of sources for confusion exist, for example if:

- The same symbol is used for multiple purposes or in multiple contexts e.g. ”–” is used to represent both subtraction and as a negative
- Multiple symbols are used for the same purpose/meaning e.g. ÷ and $\frac{\square}{\square}$ are both used to represent division
- A symbol which has been “met before” by a student in a given context can suddenly take a greater panel of meanings in a new context.

The Equals Sign: Origins

The first use of the equals sign was in 1557, by Robert Recorde, in the Whetstone of Witte.

* "to avoide the tediouse repetition of these woordes : is equalle to : I will "sette as I doe often in woorke use, a paire of paralleles, or Gemowe" [*i.e.*, twin] "lines of one lengthe, thus : ═══ : bicause noe 2 thynges can be moare equalle."
—The Whetstone of Witte, 1557.

No two things can be more equal than parallel lines of the same length.

What does it mean to our students?

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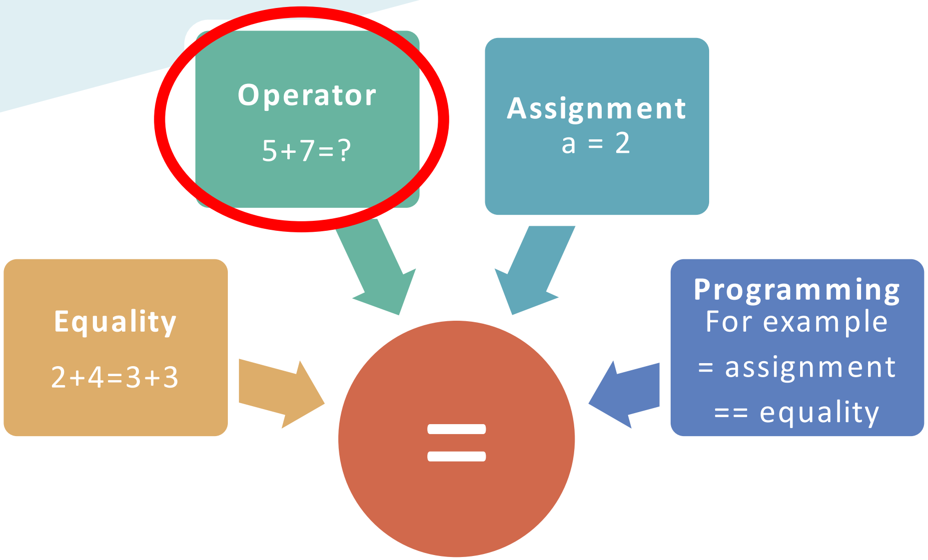
Nuances of the Equals Sign



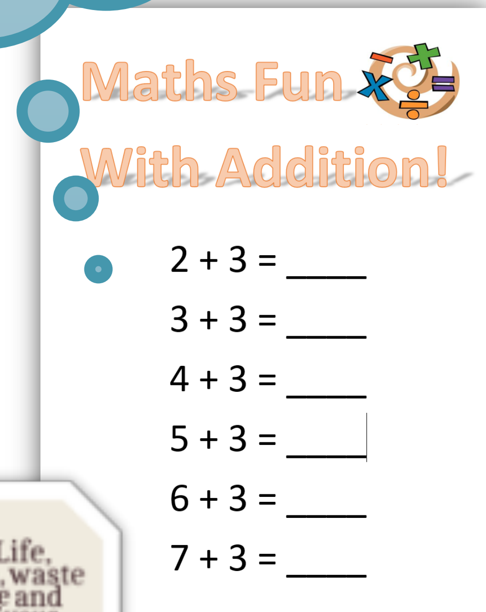
Research shows that students are only recognising the operator aspect of this symbol – **they only see an equal sign as an instruction to "do something"**

40% of the university lecturers we spoke with said their students misuse the equals sign!

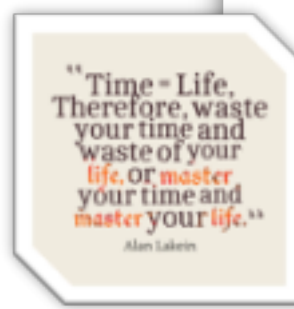
The Equals Sign

We tend to reinforce an operational rather than relational view of ‘=‘

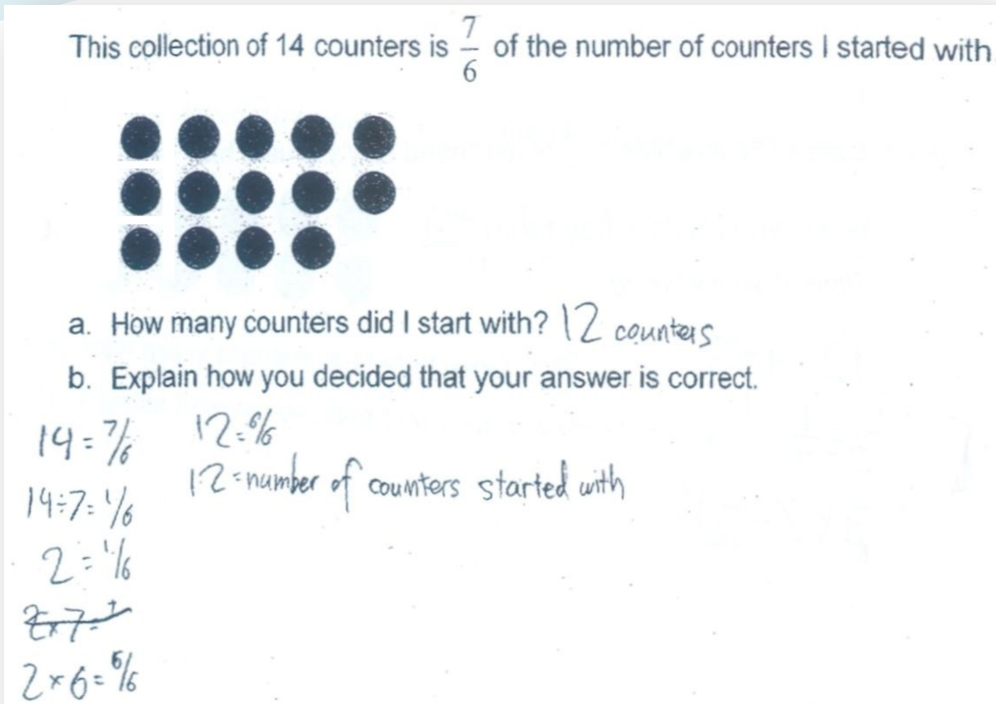
Children learn “operations are on the left” when it’s always set out like this



- There’s a strong focus on completing addition and subtraction calculations which ingrains that students must ‘write the answer’
- Many calculators use the button labelled ‘=‘ to mean ‘calculate’
- In spreadsheets, the formula =A1+B1 in cell C1 is *both* an instruction, and a way of expressing a relationship
- In slogans e.g. ‘Effort = Success’ the equals is shorthand for saying ‘leads to’, which mimics the operational interpretation in maths



The Equals Sign: Student (mis)use



This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.

a. How many counters did I start with? 12 counters

b. Explain how you decided that your answer is correct.

$14 = \frac{7}{6}$

$14 \div 7 = \frac{1}{6}$

$2 = \frac{1}{6}$

$\cancel{2 \times 7 =}$

$2 \times 6 = \frac{6}{6}$

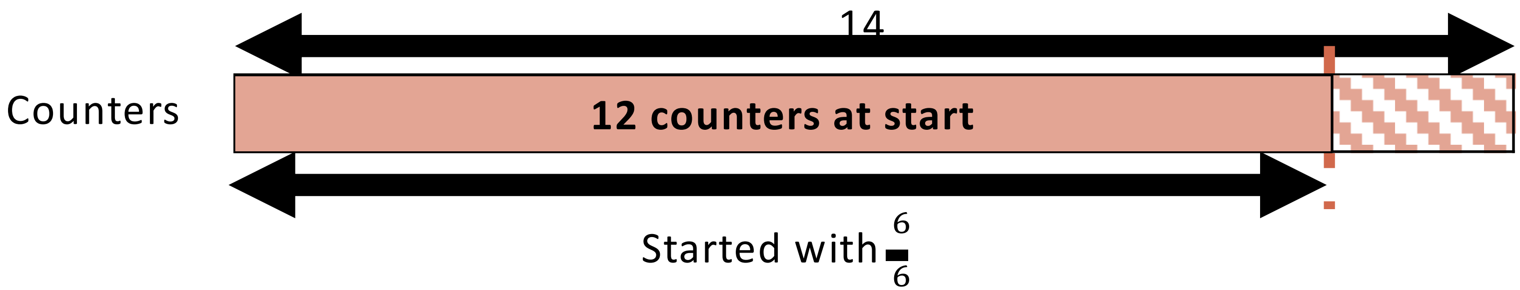
$12 = \frac{6}{6}$

12 = number of counters started with

How might they write it?

14 Counters is $\frac{7}{6}$ the number I started with. How many did I start with?

Try modelling the situation, to support writing the solution logically.

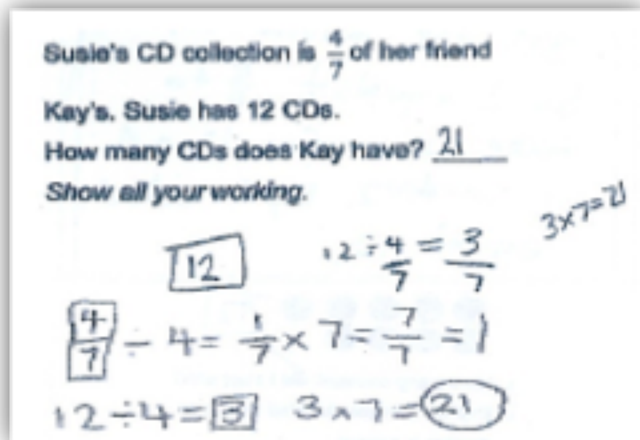


Divide the bar into 7 equal parts and call each of them “1 Unit”
7 Units = 14 counters
1 Unit = 14 ÷ 7 = 2 counters

We started with 1 whole, which is the same as $\frac{6}{6}$, or 6 Units in the model.

6 Units = 6 × 2 = 12 counters
We started with 12 counters.

The Equals Sign: Student (mis)use



Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.

How many CDs does Kay have? 21

Show all your working.

$\boxed{12}$ $\quad 12 \div \frac{4}{7} = \frac{3}{7}$ $\quad 3 \times 7 = 21$

$\boxed{\frac{4}{7}} \div 4 = \frac{1}{7} \times 7 = \frac{7}{7} = 1$

$12 \div 4 = \boxed{3} \quad 3 \times 7 = 21$

The Equals Sign: Trouble!

Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?

Common response:

2.30 + 3.20 = 5.50 - 1.50 = 4.00

Doesn't show understanding that " = " means both sides must be equivalent.

Students need to learn to set out their working for multi-step problems in a logical sequence of number sentences.

Better response:

2.30 + 3.20 = 5.50

5.50 – 1.50 = 4.00

Diagnosing Equivalence Issues

Try drawing out the issues using problems where operations are on both sides of the equation.

Arithmetic problems presenting operations to the left, e.g. 5 + 3 = 8 don't show the interchangeable nature of the two sides of an equality sign as clearly.

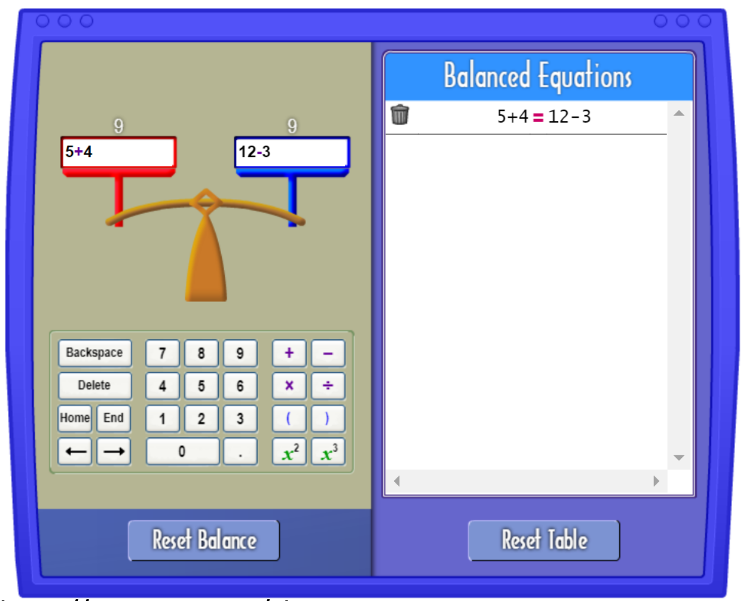
For example, with the question :

$$3 + 7 + 5 = 3 + \_\_\_$$

Many children rely on their observation that "operations are on the left side" and write "7 + 4 + 5 + 7 = __" (McNeil & Alibali, 2004)

Many children rely on their 'knowledge' that they should "perform all given operations on all given numbers" and put 23 (instead of 9) in the blank (McNeil, 2007)

Relational thinking: Balances



https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Pan-Balance----Numbers/

SCAN!

NCTM balances Activity

Some Other Ideas

Arithmetic practice can be modified to promote a **relational** view:

- ✓ Show equality independently of arithmetic, so children solidify a relational view. For example, 28 = 28.

- ✓ Include non-standard questions in practice. For example:
  - ➢ $\_\_\_ = 5+11$ [Operations on the right]
  - ➢ $7 + 6 = 10 + \_\_\_$ [Operations on both sides]
  - ➢ $2 \times \_\_\_ = 4 + 2$ [Mixture of operations on both sides]
  - ➢ $7 \times 2 + 3 - 2 = 5 \times 2 - 1 + \_\_\_$ [Multioperation arithmetic equalities]

The Multiplication Symbol

How do we introduce multiplication of whole numbers?

- Repeated addition is a common starting place
- We often then introduce language such as ‘groups of’ or ‘times’
  5 x 3: ‘5 groups of 3’ vs. **3 x 5**: ‘3 groups of 5’
  - Do they read the same?
  - Are they actually mathematically different?
  - How well does this convey the commutative laws of multiplication?
  - How might this language guide students’ future thinking?

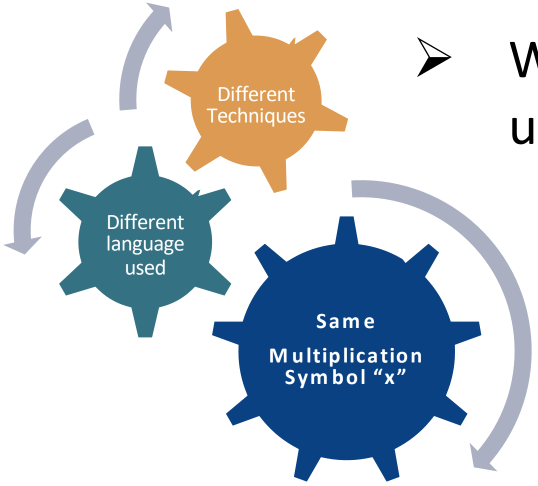
Understanding the limitations helps us to address them *before* problems set in!

The Multiplication Symbol

How might we introduce multiplication of negative numbers?

5 x -3 ‘5 groups of -3’
-3 x 5 ‘-3 groups of 5’…Hmmm

- Repeated addition? The first example could be reframed as repeated subtraction, but how about the second?
- ‘Groups of’ language only works for the first example.
- ‘Times’ language only works for the second example.
- Will students be trying to sense-make using their prior understanding of multiplication and the ‘×’ symbol?



Teaching tip: Arrays

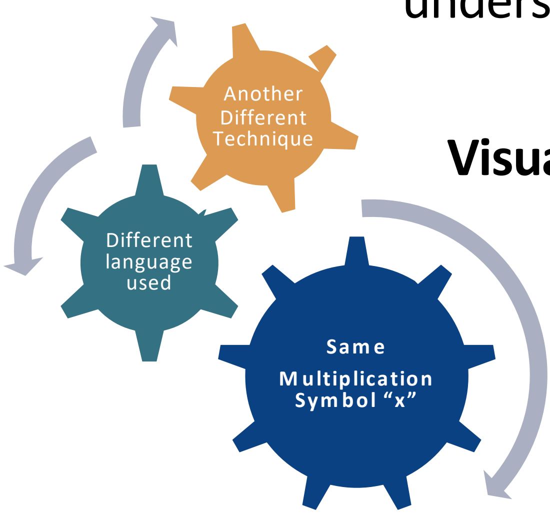
Using Arrays and then regions/area models can support students to develop a better understanding of commutativity in multiplication. This is required in many explanations of multiplication of negative numbers…

A key development in multiplicative thinking occurs when students shift from counting 'groups of' (e.g. 1 three, 2 threes, 3 threes etc.) to seeing the number of groups as a factor (e.g. 3 ones, 3 twos,3 threes, etc.) and then generalising (e.g. 3 of anything is double the group and 1 more group) (Siemon, n.d.).

The Multiplication Symbol

How might we introduce multiplication of fractions?

- Repeated addition, ‘groups of’ & ‘times’ language don’t fit
- Typically we introduce new language - ‘of’
- Again, the same symbol (‘×’) is used.
  Will students be trying to sense-make, using their prior understanding of multiplication and the ‘×’ symbol?



Visual strategies can help (e.g. Arrays or Bar Models)

Key Idea: Forge Connections!

Be **explicit** regarding the different roles of symbols when **changing contexts or meaning**.

Highlight similarities and differences in layout, techniques, and context/meaning in these instances, to help students **forge connections**

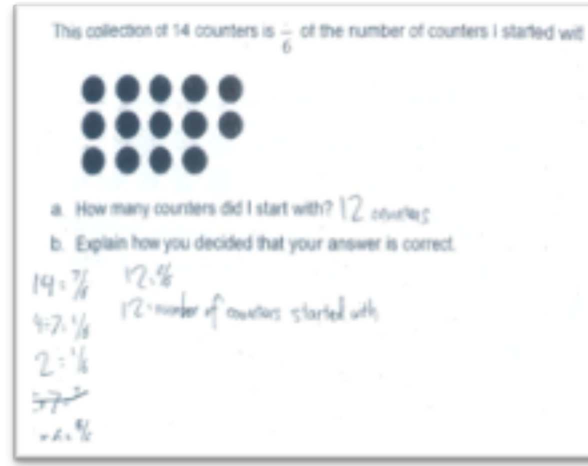
What is a Correct Solution?

Is it the correct number/symbol at ‘the end’?

Does it include a correct explanation?

What do we value?

Would you have marked the student as correct?



- When students have an incorrect final answer, do we look to acknowledge working?
- When students have a correct final answer, do we review working, or do we automatically think they have acquired the necessary skills?
- Asking students to ‘think aloud’ can be startlingly revealing at all levels.

Mathematics should make sense!

The Problem Perpetuates

We asked some university lecturers about their students' mathematics…

*"…so **for them it's all about the answers**, because so much of their work is multiple choice or short answer and it's all about whether the answer is correct.*

'I got 5, it doesn't really matter what I scrawled down on the piece of paper, I got 5, that was the correct answer',

*whereas what we want to do at university is not just make them get the correct answer, but **explain how they got it**, set it out properly and we're trying to teach them how to write mathematics, and*

***that is a huge shift even for the top students**".*

Communication

Students struggle with mathematical communication

- ***Lecturer:*** *"They just don't write down what they're doing, they don't explain. It's just literally, they think they just need to write a page of equations with each [of] these funny little symbols joining everything together and they'll think they're done"*
- ***Secondary Teacher:*** *"It makes you wonder when you're looking at it, about the students' understanding of exactly what they are doing, whether they have memorised a step or whether they actually understand exactly what it is that they have worked out"*
- ***Primary Teacher:*** *What do you see?*

Thoughts for the classroom

- Be aware that students may perceive a discontinuity in the dialogue or infuse an inappropriate prior interpretation, if different symbolic representations are used without clarification
- Carefully describe symbols and their uses as they are introduced, as they change in meaning, or as you notice a misuse of a symbol
- Be specific about the requirements for mathematical writing
- Expect and value notational rigour – don't assume 'careless' errors are not the product of misconceptions, even at an early age
- Provide opportunities for students to engage in dialogue, where they can practice using mathematical language
- Pay attention/comment on whether students' notation is correct in homework. They'll thank you in the long-run!

Some Useful Prompts

- ✓ Read your workings out loud to a friend – does it make sense?
- ✓ Read me your problem and I'll write it at the whiteboard.
- ✓ How else could we write this?
- ✓ What does this mathematical sentence mean?
- ✓ Could someone else, who had not been in this class, follow your working and use it to solve a new problem?
- ✓ Show me another way to represent and solve this problem.

Resources

- AMSI Calculate – Maths Education Research
- The key ideas and strategies that underpin Multiplicative Thinking (PowerPoint presentation download) by Dianne Siemon
- NRICH resource – Arrays
- NCTM Balances Activity

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Thank you

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